A conjecture concerning determinism and phases in quantum mechanics

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Abstract. It is shown that it is possible to introduce determinism into quantum mechanics by tracing the probabilities in the Born rules back to pseudorandomness in the absolute phase constants of the wave functions. Each wave function is muliplied by an individual phase factor $\exp(\mathrm{i}\alpha)$ with a phase constant α and represents an individual quantum system. In an ensemble of systems the phase constants are conceived to be pseudorandom numbers distributed uniformly in $[0, 2\pi]$. Based on the phase constants of both the system measured and special constituents of the measuring apparatus, a criterion is conjectured which determines the occurrence of a reduction (collapse), taken as a spatial contraction, and thus determines the outcome of any individual measurement. Nonlocality and contextuality are accounted for.

Keywords: determinism, phases, reduction, collapse, quantum measurement, Born rule, hidden variables

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Citius emergit veritas ex errore
quam ex confusione
Francis Bacon

1 Introduction

The probabilities in standard quantum mechanics are fundamentally different from those in classical (pre-quantum) statistical mechanics. In statistical mechanics the probabilities are thought of as stemming from our ignorance concerning the finer details of the physical situation considered. If we knew, for example, the position and momentum of every particle in a gas at a particular moment in time, we could calculate everything that happens at any other time. Of course, it is impossible for us to account for each one of the 10^{23} or so particles, but the idea is that this is possible in principle. Therefore, in principle, classical statistical mechanics is a deterministic theory. Any random appearance of macroscopic quantities is due to their very sensitive dependence on the microscopic initial conditions, that is, in the above example, on the values of the initial positions and momenta of the particles [1], [2].

In contrast to this, the probabilities in quantum mechanics, according to contemporary orthodoxy, cannot be attributed to any underlying, more detailed specification of the situation considered. According to this view quantum mechanics is irreparably indeterministic. As is well known, this has always been considered a serious drawback by Einstein; and Dirac wrote [3]:

> It may be that in some future development we shall be able to return to determinism, but only at the expense of giving up something else, some other prejudice which we hold to very strongly at the present time.

I think there are still many other physicists and philosophers who would be glad to see determinism established in familiar quantum mechanics too, even if only in principle, as in classical statistical mechanics, that is, even if the underlying, more detailed, descriptions could not always be completely controlled in practice [4].

Such an incorporation of determinism into quantum mechanics is suggested in the present article.

Actually there is already a deterministic quantum theory: the de Broglie-Bohm theory [5], [6]. It is however not concerned with the phases but introduces new variables into the formalism. It will be considered in Section 12.

The general procedure of making quantum mechanics deterministic is to introduce additional variables so that in principle the outcome of each individual measurement is determined, and over many repetitions the distribution of the results is in accordance with the probabilities given by the approved formulas of quantum mechanics. Such variables are usually called hidden variables or hidden parameters, terms originally coined by v. Neumann and used by him in the introduction to his book [7]. Actually some of the parameters or variables that have been considered

in the literature are not hidden at all, so terms like 'uncontrolled' [8, p. 92], 'determining' or 'fixing' variables would be more appropriate. Nevertheless, following entrenched usage, we shall also speak of hidden variables.

In the present approach the hidden variables are equated with the absolute phase constants of the quantum mechanical wave functions. Each wave function represents an individual quantum object and is multiplied by an individual phase factor $\exp(\mathrm{i}\alpha)$ with a constant phase α. In an ensemble of objects the phase constants α are thought of as being pseudorandom numbers uniformly distributed in $[0, 2\pi]$. That is, the phase constants only seem random, but in reality they are determined by certain initial conditions. This is in the spirit of chaos theory, which was systematically developed since the sixties of the past century [2, p. 971]. Then, a criterion is conjectured which determines the outcome of any individual experiment based on the phase constants of the measured system and that of a small sensitive cluster in the measuring apparatus.

The experimentally confirmed violations of the Bell inequality show that no hidden variables can exist which would lead to a local description of nature. This does not mean that the variables themselves must be nonlocal, in the sense that it would be impossible to associate them with particular spacetime regions. An example of such local variables is provided by the particle positions in the de Broglie-Bohm theory, a theory, which as a whole is nonlocal. Nevertheless, the phases considered in the present article are patently nonlocal quantities.

Another restriction comes from the Bell [8] and the Kochen-Specker [9] theorems and requires any hidden-variable theory to be 'contextual' if it is to reproduce the predictions of standard quantum mechanics. In the case of hidden-variable models contextuality means that the outcome of an experiment depends upon hidden variables in the apparatus [10]. We shall see that even with the requirement of nonlocality and contextuality it is possible to account for the experimental results when the hidden variables are equated with the absolute phase constants.

In Section 2 we describe a deterministic computer simulation of quantum effects in a special case, which influenced our treatment of the general case in the following sections. In Section 3 a popular argument against the very existence of a definite phase is refuted. In Section 4 we quote remarks on the phases made in Born's original papers and in the work by Ax and Kochen, and we compare our proposal with determinism in classical statistical mechanics. In Section 5 we consider the role of the absolute phase in superpositions, transformations, and in the reduction process.

In Sections 6 to 8 it is argued that the measurement of any physical quantity, by means of known deterministic physical laws, classical or quantum, can be explained by a spacetime position measurement or a sequence thereof. In this the occurrence of a reduction (collapse) in the form of a spatial contraction is an essential component.

Section 9 defines the Born probability rules. In Section 10 a criterion is formulated which decides whether or not in the individual situation a reduction/contraction will occur which at a certain position leads to a macroscopic effect. The crucial new ingredient is that the phase constants of both the system measured and part

of the measuring apparatus are involved. Section 11 shows how and under what approximations the criterion leads to the Born rules.

Finally in Section 12 our conjecture is compared with the de Broglie-Bohm theory.

2 Computer simulation of quantum effects

As a special concrete example we consider the gradual emergence of an interference pattern from point densities on a screen behind a wall with a double slit. A source emits electrons one by one, and on the screen points are produced. As the points accumulate, an interference pattern emerges. Fig. 1 shows a sequence of intermittent snapshots taken from a computer simulation [11], and Fig. 2 shows the similarity to the results of a physical experiment [12].

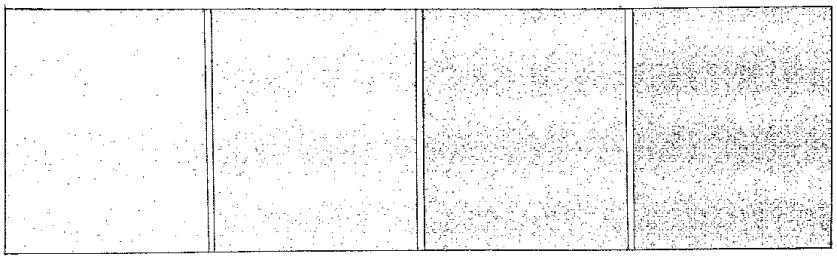

Figure 1: Computer simulation of the event-by-event buildup of an electron interference pattern behind a wall with a double slit [11].

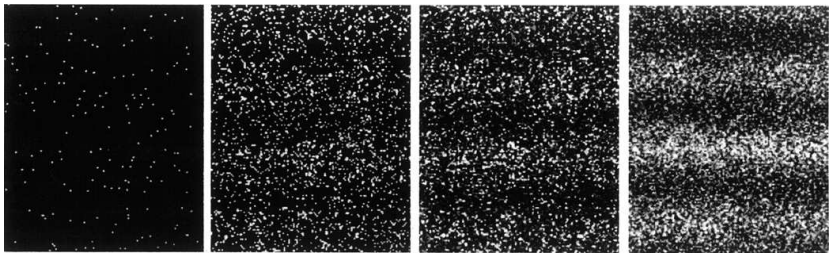

Figure 2: Physical experiment on the buildup of an interference pattern behind an electron biprism [12].

Let us consider the simulation more closely. It proceeds in four steps:

(1) The screen is subdivided into a finite number of small areas of equal size. To (the centre of) each area an x- and a z-coordinate, both in the interval $[0, 1]$, are attributed.

(2) A number X is calculated by an appropriate algorithm which produces a series of numbers with a uniform random distribution in the interval $[0, 1]$, starting

from some arbitrarily chosen initial number; or, X is taken from a table of random numbers calculated in advance [13, p. 63-65]. X refers to the x-coordinate of a particular area.

(3) In the same way a second random number Z is chosen, which refers to the z-coordinate of that area.

(4) The third random number R is the important one. It determines whether or not a point is set in the area determined by X and Z: if

$$\eta \, \frac{|\psi(x,z)|^2}{|\psi|^2_{\max}} \geq R \tag{2.1}$$

a point is set, or else no point is set, and the procedure starts again with step (2).

$\eta \leq 1$ is a proportionality factor, the same for all areas, which characterizes the degree of sensitivity of the areas: if, for instance, $\eta=0$, no point is ever set. $\psi(x,z)$ is the (Schrödinger) wave function at the point (x,z) on the screen, calculated for the particular physical situation under consideration. $|\psi|^2_{\max}$ is the maximum value of $|\psi(x,z)|^2$ over the sensitive region.

It is easy to see that the criterion (2.1) leads to Born's rule in the form

$$p \propto |\psi(x,z)|^2. \tag{2.2}$$

Indeed, every random number which lies in the interval [0, $\eta|\psi|^2/|\psi|^2_{\max}$] leads to a point. The probability that a random number with uniform distribution will lie within this interval is just $\eta|\psi|^2/|\psi|^2_{\max}$, so the point density is proportional to $|\psi(x,z)|^2$ (cf. [13, p. 71]).

The random number R determines the outcome of the 'experiment': point or no point. Thus, from the moment the value of this random number is known, everything is determined. Actually, and this is the point, the random numbers described are not random but *pseudorandom*; they are really determined by the initial number chosen and the algorithm applied. So here we have complete determinism. The problem is to find *physical* quantities that can play the role of the pseudorandom numbers and lead to the Born rules in the general case.

3 Valuation of the absolute phase

In our proposal the physical quantities that play the role of the pseudorandom numbers are the absolute (global, overall, spacetime-independent) phase constants of the wave functions representing the quantum objects.

Before explaining this in detail we want to deal with the frequently encountered assertion that the absolute phase of a wave function is undetermined if the wave function represents a definite number of particles, in the same way as the position is undetermined if the wave function represents a particle of definite momentum. This would contradict our approach, in which a definite phase is ascribed to every wave function.

We reject that assertion. It stems from the introduction of a phase operator, which does not commute with the particle-number operator. There are, however,

serious difficulties with the construction of a phase operator. The original phase operator, introduced by Dirac [14], is not Hermitian and thus cannot be an observable. Subsequently there have been many attempts to construct a phase operator that is less deficient, but each proposal had its own difficulties and none has met with general approval. Details can be found in [15] and the literature cited there.

Moreover, in a plane wave $\exp[\mathrm{i}(\boldsymbol{kr} - \omega t + \alpha)]$ the phase constant appears on an equal footing with the time t. So, what holds for time should also hold for phase. Regarding time, Pauli [16] pointed out discrepancies in the commutation relations between the operators time and energy; he wrote:

> We, therefore, conclude that the introduction of an operator t is basically forbidden and the time t must necessarily be considered as an ordinary number ("c-number") in Quantum Mechanics.

Indeed, there is a position operator but no time operator in non-relativistic quantum mechanics. And in relativistic quantum field theory both position and time are parameters, not operators. We therefore take the view that we do not need a phase operator for observing the phase, just as we do not need a time operator for observing the time. Dispensing with the phase operator frees us from the phase/number uncertainty relation and allows us to ascribe a definite phase to every wave function, though the actual values of the phases may be unknown to us.

Our position is supported experimentally by the beats and interference fringes that can be observed when two independent quasi-monochromatic laser or maser beams (photon wave functions) are superposed [17], [18], thereby disobeying Dirac's ban on interference between two different photons [19]. In the superpositions the absolute phases of the individual wave functions become relative phases and determine, for example, the positions of the interference fringes. Note that it is not possible to speak of a definite phase difference between two independent wave functions if these functions do not each have a definite phase of their own. After all, an absolute phase is as relative to some standard phase as an absolute mass is relative to some standard mass (cf. also [20, p. 41]).

In our theory the absolute phases are real physical quantities. Indeed, some time ago I worked out a realist interpretation of the mathematical formalism of quantum mechanics in which the concept of a point particle is replaced by the concept of a real, objective wavepacket [21], [22], [23]. The argument $\boldsymbol{r}$ in the wave function $\psi(\boldsymbol{r}, t)$ does not mean the position of the (point)particle. The position of the wavepacket is rather given by the parameter $\boldsymbol{r}_0$ in $\psi(\boldsymbol{r}, t)$ which specifies its centre. I will not go into a description of that interpretation here, but the arguments it puts forward are reflected in the fact that the concept of a point position is never used in the present article and that I mostly speak of a wavepacket or wave function rather than of a particle that undergoes an interaction or causes an effect. Readers who are still determined adherents of the Copenhagen interpretation can easily translate my formulations into the language they prefer.

4 The absolute phase as a pseudorandom number

The idea that the phase has something to do with the probabilistic feature of quantum mechanics is not new. Indeed, the fact that in the expression (2.2) the phase of ψ does not appear at all, immediately raises the question whether this might not be the very reason for the probabilistic feature. Actually, Born raised that question already in the first two papers in which he proposed the probability interpretation [24, p. 826, 827], [25, p. 866], although only briefly and without pursuing the matter further. In [25, p. 866] he wrote:

> ... we have so far no reason to believe that there are some inner properties of the atom which condition a definite outcome for the collision. Ought we to hope later to discover such properties (like phases of the internal atomic motion) and determine them in individual cases?

A recent treatment is the paper by Ax and Kochen [20], where the authors state in the abstract that

> In the new interpretation, rays in Hilbert space correspond to ensembles, while unit vectors in a ray correspond to individual members of such an ensemble. The apparent indeterminism of SQM [statistical quantum mechanics] is thus attributable to the effectively random distribution of initial phases.

Ax and Kochen's elaboration of the idea has a strong mathematical orientation. It is not deterministic and differs in several other respects from the rather physically oriented elaboration in the present article. But evidently Ax and Kochen consider the equating of the phases with the hidden variables not to be a priori forbidden by the Kochen-Specker contextuality theorem [9], [26].

We thus consider every wave function of the standard formalism of quantum mechanics to be multiplied by an individual phase factor $\exp(\mathrm{i}\alpha)$, with α being a real number, independent of $\boldsymbol{r}$ and t. The wave functions are still normalized to 1. In an ensemble of such wave functions the αs are pseudorandom numbers. They are assumed to be uniformly distributed in the interval $[0,\, 2\pi]$ if they refer to some kind of equilibrium, that is, if we do not select sets of wave functions with determined phase values and then isolate them (if this is possible; see below). This cannot be proved but is a postulate. It fits with the postulate of random phases already met in quantum statistical mechanics [27, p. 173, 190]. And it is analogous to the postulate in classical statistical mechanics that in equilibrium the density of points in phase space is uniform [27, p. 129]. In other words, probabilities in quantum mechanics will be traced back to uniform distribution of phases in the same way as probabilities in classical statistical mechanics are traced back to uniform distribution of points in phase space.

The total wave function written in polar form reads

$$\psi(\boldsymbol{r},t) = e^{\mathrm{i}\alpha}\, e^{\mathrm{i}\varphi(\boldsymbol{r},t)}|\psi(\boldsymbol{r},t)| = e^{\mathrm{i}(\alpha+\varphi(\boldsymbol{r},t))}|\psi(\boldsymbol{r},t)|,$$

where $\exp(\mathrm{i}\varphi(\boldsymbol{r},t))\,|\psi(\boldsymbol{r},t)|$ is a normalized solution of the Schrödinger equation. $\alpha+\varphi(\boldsymbol{r},t)$ is the total phase, and α is the absolute phase constant. If α is uniformly distributed in $[0,\,2\pi]$ then $\alpha+\varphi(\boldsymbol{r},t)$ modulo 2π for any fixed $\boldsymbol{r}$ and t is also so distributed [28, Vol. II, p. 61-65]. Thus, with respect to random appearance the total phase and the phase constant play the same role, and it is irrelevant which moment we choose to be the initial moment for defining the phase constant.

Is it possible to determine the phase constant of a wave function experimentally? In the case of independent quasi-monochromatic laser beams considered in Section 3 experimental determination may be possible if each beam is a single condensed wavepacket representing many photons, which all have the same phase. Then the phase constant of a particular beam may be declared the standard phase constant, while those of the other beams can be determined relative to that standard phase constant by means of interference fringes. In producing the interference fringes some photons will disappear, but still enough photons may survive to form the beams with the original phase constant. Such a procedure may also be possible with massive bosons, but it is not possible with fermions as there are no condensed fermionic wavepackets, and the problem is left open here. This is different from classical kinetic theory.

Note, however, that even if we were not able to experimentally determine the absolute phase of a given wave function, this does not mean that it would not make sense to attribute a definite though unknown absolute phase to it. Compare this with the procedure of attributing definite positions and momenta to the individual particles of an ensemble in classical kinetic theory. Or compare it with Feynman's example of the concept of the inside of a brick, which no one can ever see [29] (cf. also [20, p. 41]).

5 Changes in the absolute phase constant?

The Schrödinger equation (or any of the quantum mechanical equations of motion) is linear and homogeneous and therefore provides no information concerning a phase factor $\exp(\mathrm{i}\alpha)$ in front of its solutions, and so long as temporal evolution is completely governed by the Schrödinger equation, the absolute phase constant is indeed independent of space and time and remains the initial phase constant. We will consider a few cases in detail:

What happens to the phase constants when a one-particle wavepacket ψ is written as $\psi=\sum_i \lambda_i\psi_i$? The ψ_is here are simply terms in a largely arbitrary mathematical decomposition. As the phase constant refers to the wavepacket that represents the quantum object as a whole, it is put in front of each of the ψ_is. If the ψ_is were to contain individual random phase factors $\exp(\mathrm{i}\alpha_i)$, then, after averaging over the α_is, there would be no interference terms in the probability expressions (like $|\psi|^2$, for example), and the superposition could only describe a statistical mixture (cf. [30, p. 254]).

Multi-particle wave functions representing particles which are independent of each other can be written as a product of single-particle wave functions. The total phase constant is then the sum of the individual single-particle phase constants.

When the particles (i.e. wavepackets) become entangled, the total wave function will turn into a sum (or series or integral) of product wave functions which can no longer be written as a single product:

$$\Psi = \sum_{i,j} c_{ij}\, \psi_i^{(1)} \psi_j^{(2)} \tag{5.1}$$

with $c_{ij} \neq a_i b_j$. In accordance with what we have said above concerning the one-particle wavepacket, single-particle contributions $\psi_l^{(k)}$ for fixed k still have the same phase factor $\exp(\mathrm{i}\alpha^{(k)})$ for all l, so that each term in the sum (5.1) has the factor $\exp(\mathrm{i}[\alpha^{(1)} + \alpha^{(2)}])$, which can be put in front of the sum. This applies also when the particles are identical and Ψ is an (anti)symmetric function

$$\Psi = \sum_{i,j} c_{ij} \left(\psi_i^{(1)} \psi_j^{(2)} \pm \psi_j^{(1)} \psi_i^{(2)} \right).$$

Thus, regardless of whether the particles are entangled or not, the phase constant of the total wave function is the sum of the phase constants of the individual-particle wave functions.

What happens to the absolute phase constants in transformations in Hilbert space which correspond to transformations in classical physical space, such as rotations or Lorentz transformations? In the traditional ray representation of the wave functions, where their absolute phases are given no physical significance, the transformation operators in Hilbert space also only need to be ray representations, that is, they only need to be defined up to an arbitrary phase factor. Instead of writing the transformation T in the form

$$\psi_1 = T\, \psi_2$$

it is therefore possible to write

$$\psi_1 = e^{\mathrm{i}\alpha}\, T\, \psi_2 = T\, e^{\mathrm{i}\alpha}\, \psi_2 = T\, \psi_3.$$

However, this means that the wave function ψ by a mere transformation may acquire an additional arbitrary contribution to its absolute phase constant. As our theory is based on a vector representation of the wave functions, where their absolute phases do have physical significance, we must also insist on a vector representation of the transformation operators. That is, if in a ray representation the product of two operators is written as $T_k = \exp(\mathrm{i}\alpha_{ji}^k)\, T_j\, T_i$, where α_{ji}^k is a certain phase depending on the phases for T_i, T_j and T_k, in a vector representation the phases have to be fixed in such a way that all of the α_{ji}^ks are zero, $T_k = T_j\, T_i$, and the transformations are not equivalent with attaching arbitrary phase factors to the wave functions. This is not possible for all transformations, for example not for Galilean boosts, but it *is* possible for the important cases of permutations, Lorentz boosts (pure Lorentz transformations), space and time translations, and also for spatial rotations, with

the restriction that the phase fators $\exp(\mathrm{i}\alpha^k_{ji})$ for spinors are either $+1$ or -1, rather than only $+1$ [31], [32, Chapter XV, §6–§8], [33], [34].

Now, besides the Schrödinger equation there is the reduction, and what is a phase constant in the Schrödinger theory need not necessarily be a constant in the reduction process. We stipulate, for the time being, that even in the reduction process the phase constant remains unchanged. Other assumptions, which would also leave the phase constants uniformly distributed and which could also lead to the Born probability rules, are a constant phase shift or the addition of another pseudorandom number in $[0, 2\pi]$ modulo 2π. The constancy seems to be the simplest assumption which makes the theory in principle fully deterministic.

6 The spacetime nature of measurements

Our conjecture will be based on a conception of a quantum mechanical measurement that is different in several respects from most other treatments. There is an abundance of literature on the measurement process in quantum mechanics [35] and it is not our purpose to review it here. We restrict ourselves to those aspects that are relevant to our approach. One point of difference is that we take the view that it is *always* possible to relate a property of a system by deterministic physical laws to spacetime events, so that the measurement of any property is ultimately explained by one or a sequence of spacetime position measurements. This is not a new idea. Einstein, for one, wrote [36]:

> Now it is characteristic of thought in physics, as of thought in natural science generally, that it endeavours in principle to make do with "space-type" concepts *alone*, and strives to express with their aid all relations having the form of laws. The physicist seeks to reduce colours and tones to vibrations, the physiologist thought and pain to nerve processes, in such a way that the psychical element as such is eliminated from the causal nexus of existence, and thus nowhere occurs as an independent link in the causal associations.

And similar statements have been made by Bell [8, p. 10, 34, 166] and many others, for example [37, p. 350, 404], [38]. In our treatment it plays a fundamental role. The measurements we are concerned with are the following three-step processes:

In the first step
the *active region* of the measuring apparatus must achieve that the incoming wavepacket $\psi_{\mathrm{i}}(\boldsymbol{r}, t)$, representing the incoming particle, develops into a superposition

$$\psi_{\mathrm{s}}(\boldsymbol{r}, t) = \sum_n c_n \, \psi_n(\boldsymbol{r}, t) \tag{6.1}$$

with
$$c_n = (\psi_n(\boldsymbol{r}, t), \psi_{\mathrm{s}}(\boldsymbol{r}, t)) = \int \psi_n^*(\boldsymbol{r}, t) \, \psi_{\mathrm{s}}(\boldsymbol{r}, t) \, \mathrm{d}^3 r \tag{6.2}$$

or, in the continuous case,

$$\psi_{\rm s}(\boldsymbol{r},t) = \int c(a)\,\psi(a;\boldsymbol{r},t)\mathrm{d}a \tag{6.3}$$

with $$c(a) = (\psi(a;\boldsymbol{r},t),\psi_{\rm s}(\boldsymbol{r},t)) = \int \psi^*(a;\boldsymbol{r},t)\,\psi_{\rm s}(\boldsymbol{r},t)\,\mathrm{d}^3r \tag{6.4}$$

with normalized eigenpackets $\psi_n(\boldsymbol{r},t)$ and $\psi(a;\boldsymbol{r},t)$, respectively, of the observable corresponding to the physical quantity under consideration,

subject however to the condition that the different eigenpackets are located in different regions of space.

This "preparation step" is not a mathematical but a physical problem. The design of an appropriate apparatus and with it the choice of the corresponding Hamilton operator $H_{\rm M}$ (or temporal evolution operator $U_{\rm M}$, such as $U_{\rm M}(t,t_0) = \exp[-\mathrm{i}(t-t_0)H_{\rm M}/\hbar]$, the index M standing for "measurement") is a challenge to the inventiveness and ingenuity of the experimental physicist. He or she is usually guided by the idea of an apparatus that would be used if the quantity considered were classical (some kind of correspondence principle?).

In the second step
the superposition $\psi_{\rm s}(\boldsymbol{r},t)$ of spatially separated eigenpackets enters the *sensitive region* of the measuring apparatus, which we will call the **screen**, and at a small **cluster** in the screen undergoes a reduction, that is, a **contraction** to a wavepacket with the spatial dimensions of the cluster, provided the criterion formulated in Section 10 is met at that cluster.

The cluster may be a group of strongly bound atoms or molecules, one single molecule, or only one single atom, depending on the particular type of the screen employed (more on this in Section 7). We shall from now on speak of contraction rather than reduction or collapse.

Though the form of the wavepacket after the contraction is not required for the explanation of the Born rules, we want to specify it in order to be close to the usual postulates. These, we recall, are [30, Sections 3.2.3 and 3.5.2]:

(1) In the case of a discrete eigenvalue spectrum of the observable, the wave function that leaves the measuring apparatus is the eigenfunction associated with the eigenvalue measured, and

(2) in the case of a continuous spectrum the escaping wave function is a normalized integral of (improper) eigenfunctions extended over the range of eigenvalues determined by the finite resolving power of the apparatus used. – We always restrict ourselves to non-degenerate eigenvalues; the case of degenerate eigenvalues is only technically more complicated but adds nothing essential.

Our procedure is somewhat different because we want to treat the cases of discrete and continuous spectra on the same footing. Thus we conceive that for both discrete and continuous spectra the fanned out wavepacket $\psi_{\rm s}$ in the reduction contracts to the spatial dimensions of the cluster (cf. e.g. [8, p. 119]). More precisely, the contracted wavepacket $\psi_{\rm c}(\boldsymbol{r}_0;\boldsymbol{r},t)$ is given by

$$\psi_{\rm c}(\boldsymbol{r}_0;\boldsymbol{r},t) = \frac{1}{N}\psi_{\rm s}(\boldsymbol{r},t)\,|(\psi_{\rm cl}(\boldsymbol{r}_0;\boldsymbol{r},t)|^2, \tag{6.5}$$

where N is a normalization factor and $\psi_{\rm cl}(\boldsymbol{r}_0;\boldsymbol{r},t)$ is the wavepacket which represents the cluster, with centre at $\boldsymbol{r}_0$. It is in priciple known. The function $\psi_{\rm c}(\boldsymbol{r}_0;\boldsymbol{r},t)$ is the normalized piece of $\psi_{\rm s}(\boldsymbol{r},t)$ which is cut out by the cluster. It is, however, not a cut with sharp edges because $\psi_{\rm c}$ is a wave function, and with a sharp cut the momentum operator defined for such a function would not be self-adjoint. If the cluster is larger than the wavepacket, formula (6.5) does not mean a contraction in the strict sense because the wavepacket $\psi_{\rm c}(\boldsymbol{r}_0;\boldsymbol{r},t)$ is then left essentially unchanged. We want to include this limiting case in the definition of contraction.

We emphasize, however, that we are only concerned with typical quantum mechanical measurements, that is with those where the spatial resolving interval of the measuring apparatus (error interval, reciprocal resolving power) is smaller than the the size of the wavepacket $\psi_{\rm s}$ representing the measured object at the moment when it meets the screen. The resolving interval is determined by the size of the clusters in the screen and, in the case of a discrete eigenvalue spectrum, by the distances between the eigenfunctions ψ_n. In addition, in our approach the clusters which are of use in the measurement must be so small that ψ_n or $\psi_{\rm s}$ over the region of the cluster can be taken to be linear functions of space. That is, a screen must be used in which the size of the clusters is small compared to all other relevant distances.

According to the Heisenberg relations the contraction to the size Δx of the cluster may increase the extent Δp_x of the wavepacket. For the minimum size $\Delta x = 0.5\times10^{-10}$ m (Bohr radius), it is $\Delta p_x \geq 10^{-24}$ kg m s^{-1}. As far as I could check, this does not conflict with experimental findings; for example, for the 50 keV electrons used in Figure 2 this means $\Delta p_y/p_y = \Delta\lambda/\lambda = 0.009$.

In the theory of Ghirardi, Rimini and Weber [39], [40] the reduction (collapse) is also a contraction, but to a spatial region determined by a new constant of nature suggested to be of the order of 10^{-7} m. Moreover, like the other collapse theories, [41], [42], that theory is not deterministic.

In contrast to the usual postulate the narrow contracted packet $\psi_{\rm c}$ of our theory may not be exactly but only approximately equal to the eigenpacket ψ_n associated with the position $\boldsymbol{r}_0$. Moreover, $\psi_{\rm c}$ is not the final wavepacket, which leaves the apparatus, because there is one more step.

In the third and last step
the contracted wavepacket $\psi_{\rm c}(\boldsymbol{r}_0;\boldsymbol{r},t)$ interacts with the wavepacket $\psi_{\rm cl}$ that represents the cluster. This is a dynamical interaction governed by the Schrödinger equation and leads to the final wavepacket $\psi_{\rm f}(\boldsymbol{r},t)$.

The final wavepacket $\psi_{\rm f}$ may be absorbed (photons, neutrons, atoms), or it may escape the interaction region. If it escapes, it may be close to the contracted packet $\psi_{\rm c}$ or it may differ appreciably from it, depending on the details of the interaction. As $\psi_{\rm c}$ may not be exactly equal to the eigenpacket ψ_n of the observable, so also $\psi_{\rm f}$ may not be exactly equal to ψ_n. It is, however, reasonable to assume that $\psi_{\rm f}$ immediately after the interaction is also, like $\psi_{\rm c}$, concentrated about the position $\boldsymbol{r}_0$ of the cluster. Therefore, though we cannot generally say that when the measurement is over, the incoming superposition of eigenpackets has turned into a function that is

exactly equal to an eigenpacket ψ_n of the observable, we may suppose that usually this happens to an acceptable degree of approximation (cf. [22, Section 4.2], [23, Reference [6]]). In a sense this goes some way towards the modal interpretation, where there is no eigenvalue-eigenstate link at all [43].

Actually, there are many thought-experiments and discussions about the wavepacket when it leaves the measuring apparatus, but to my knowledge there is no real experiment directly concerned with this question.

The dynamical interaction of the contracted wavepacket $\psi_{\rm c}(\boldsymbol{r}_0;\boldsymbol{r},t)$ with the cluster may or may not initiate the formation of a stable macroscopic effect. In a measurement the interaction must of course be such that it accomplishes this. This requires *sensitive* clusters, that is those with special properties. A typical case is an atom which is ionized, excited or de-excited and from which an avalanche develops, which involves more and more particles ("secondary electron multiplier", for example) so that the original interaction event is magnified until a spot emerges which a human eye can perceive. Generally, the magnification still involves a microscope or a chain of apparatuses with a great deal of electronic equipment [44]. In whatever form the stable macroscopic effect is realized, we shall always call it a **spot**. The formation of the spot in principle (though not in practice) completes the measurement. It is not necessary for someone to take cognizance of the spot; it is sufficient that someone *could* do so. The details of spot formation are beyond the intended scope of the present article.

The three steps may thus be summarized as:

$\psi_{\rm i} \to \psi_{\rm s} \to \psi_{\rm c} \to \psi_{\rm f}$, where

$\psi_{\rm i}$ =incoming wavepacket

$\psi_{\rm s}$= superposition of spatially separated eigenpackets ψ_n,

$\psi_{\rm c}$= contracted wavepacket, and

$\psi_{\rm f}$= escaping final wavepacket.

Note that it is only when it is embedded in a specifically designed apparatus that the position of the spot acquires meaning and can reveal some properties of the wavepacket under consideration. Also, any position measurement can determine the position at most to within the finite region of a spot, that is, of another wavepacket. What is under consideration is not a space interval but the direct measurement of the absolute localization. It is only because the spots are small compared with the extensions of the wavepackets that they can effectively be considered as points; and this, together with the fact that the wavepacket can never make two separated detectors click simultaneously, has certainly contributed to the association of a point particle with the wave function in traditional quantum mechanics. These are rather trivial remarks, but I think they are justified when one is trying to break the evil spell cast on this part of quantum mechanics.

7 The structure of the screen

In the first step of the measurement, where the apparatus according to $H_{\rm M}$ fans out the incoming wavepacket into a superposition of spatially separated eigenpackets (Section 6), there is no back action of the incoming wavepacket on the apparatus.

In the second and third steps the superposition interacts with the screen, and both the superposition and the screen are changed: the superposition contracts and the contracted packet interacts dynamically with the sceen, a macroscopic body, so as to produce a spot. We want to describe this dynamical interaction as an interaction between two quantum mechanical systems and therefore have to specify the wave function which represents the screen. This is another point of difference between our and most other conceptions of a measuring apparatus.

In conceiving a wave function which is to represent the screen, and indeed any macroscopic body, the following facts should be taken into account:

When a particle hits the screen only a small region of the screen will be affected. The processes of ionization, excitation and de-excitation can be ascribed to small sensitive clusters, and the developing avalanche does not extend over the whole screen but only over a small part of it. This is comparable to an array of little lamps one of which is switched on by the incoming wavepacket. If this were not so, neither the different beams leaving the Stern-Gerlach magnet (Section 8) nor the electrons in Tonomura's experiment (Figure 2) could produce the little spots at the different places in the screen. Quite similar situations are encountered in electron hole production, deposited heat, and pair as well as multiple particle production [44]. Also, in calculating scattering amplitudes it is an approved approximation that each scattering centre in the target material acts as if it were alone [32, p. 370].

These observations lead us to the conclusion that

(1) It is primarily the part of the wave function which describes the internal structure of the screen (the internal wave function, for short), and not the centre-of-mass wave function of the screen as a whole which is relevant in the treatment of the interaction of the incoming particle with the screen.

(2) The internal wave function in first approximation is a product of wave functions representing the constituent atoms, but in second approximation these atoms interact with each other; after all they stick together to form the screen.

Homogeneous macroscopic bodies like the screen are not homogeneous down to the atomic level. This holds only for the very special case of pure (perfect) monocrystals. Usually, macroscopic bodies are homogeneous only in a coarse-grained sense: they consist of subsystems of various types and sizes, such as molecules of different structure, composition and size, tightly bound clusters (like the grains in a photographic emulsion), micro-crystals, regions of condensation, etc.

These subsystems permanently interact with each other. There are fluctuating processes of dynamic coupling and decoupling caused by thermal excitations, there are processes of condensation and decondensation of identical wavepackets, and there is entanglement and disentanglement. The disentanglement may either be exact or only effective (more on the latter below). The exact disentanglement is the reverse of entanglement and must, in principle, exist because, like entanglement, it arises from Schrödinger dynamics, which is time-reversal invariant. Last but not least there are the ubiquitous contraction processs (reductions) between the subsystems.

The whole screen is thus not a static but a stationary aggregate of clusters, and different clusters can be assumed to have different phase constants. And any macro-

scopic body must basically be described by means of a wave function which rapidly varies with time on a microscopic scale. On the macroscopic scale the properties of the body may appear time-independent because measurements on this scale mean averages over time intervals that are large on the microscopic though small on the macroscopic scale, so that for repeated incoming wavepackets the screen each time is different even if for us huge human beings it appears always the same.

With respect to the characterisation of the screen we regard the clusters in it already as effectively classical objects. That is, we take localization due to decoherence already to apply to these clusters, rather than only to macroscopic objects. In the usual decoherence theories [45] - [49] the contraction of the centre-of-mass function of a macroscopic object as a whole, that is, its localization, is brought about by taking the unavoidable entanglement of the object with its environment into account when considering properties which refer to the object alone (performing a partial trace, over the environment, of the density matrix of object+environment). Here we extend this to *internal decoherence*, that is, we regard the 'macroscopic object' to be a cluster in the screen, and the 'environment' to be the other clusters and, in fact, the rest of the screen [50].

If the spread in the centre-of-mass function of each single cluster is σ, the coordinate of the centre of mass of a macroscopic body, consisting of, say N^3 clusters, must lie in an interval of linear dimension $\sigma/\sqrt{N}$. With $\sigma = 10^{-9}$ m, $N = 10^2$ this is 10^{-10} m. This value of σ is small compared with the experimental error interval of the usual apparatuses for measuring length. This is then no longer a typical quantum-mechanical measurement as defined in Section 6, and the length here may be considered a classical property of the body [51]. The justification for treating the internal structure of a particular macroscopic body as a classical aggregate thus depends on its particular type of internal structure. These questions certainly deserve closer study, but we will not pursue them in the present article.

Usually the quantum mechanical measurement process is described by formulas such as

$$\psi\,\Phi_0 = \Big(\sum_n c_n\,\varphi_n\Big)\Phi_0 \;\to\; \sum_n c_n\,\varphi_n\,\Phi_n \;\to\; c'_k\,\varphi_k\,\Phi_k. \tag{7.1}$$

Here ψ is the incoming particle, Φ_0 is the wave function conceived to represent the apparatus before the measurement, and φ_n is an eigenfunction of the observable O. The first arrow means that the incoming particle gets entangled with the apparatus, due to the unitary Schrödinger evolution. The functions Φ_n represent so-called "pointer states" of the apparatus [45]. Decoherence changes the superposition of the purported pointer states into an (improper) mixture, but it cannot explain why only one single pointer state is experienced [45, Section 5], [46], [49, Sections 3 and 6]. The second arrow indicates the reduction leading to one particular term of the second sum over n. The coefficient c'_k differs from c_k only because the final state $c'_k\varphi_k\Phi_k$ is again normalized to 1.

In our approach the term between the arrows in formula (7.1) is missing. The contraction leads to one particular state Φ_k of the apparatus, namely an apparatus

with a spot at a particular place on the screen. This is a pointer state, if one sticks to this concept. These pointer states are not, however, such as exist before the contraction and afterwards all but one of them disappear.

We would generally assume that when the incoming particle is a composite object, such as an atom or a complex molecule, it is its centre-of-mass wave function (as opposed to the *internal* function, which describes the internal structure of the object), that takes on the form of a narrow function at the moment when the object is formed or when it is contracted, and that this function then spreads out according to the familiar formulas. This assumption is not incompatible with the interference patterns observed with atoms and complex molecules at diffraction gratings [52], [53]. These experiments show that what is responsible for the interference patterns is the centre-of-mass function of the atom or molecule, that is, a wavepacket centred about the de Broglie wavelength $\lambda = h/m_0 v$, where m_0 and v are the mass and the velocity respectively of the atom or molecule as a whole. Numerical estimates show that the initial transverse size of this function, when it is equal to the size of the internal wave function of the atom or molecule or to the size of the cluster in the contraction, is small compared with the slit separation, and thus cannot produce interference effects. But due to spreading on the way from formation through vacuum to the grating, the transverse size of the function becomes larger than the slit separation and then does produce the interference pattern.

One might expect that in an interaction which effects a contraction, the centre-of-mass functions of both the incoming object and the cluster in the screen are contracted. The cluster, however, is already contracted to the order of its internal wave function due to its permanent interactions with the other matter of the screen, reflecting the basic asymmetry between measuring apparatus and measured object.

8 Examples

We here consider two examples illustrating our view on the spacetime nature of measurements: the eigenvalue spectrum of the observable is (a) discrete or (b) continuous.

An example of case (a),
concerned with eigenfunctions of a discrete spectrum, is the well-known Stern-Gerlach experiment: in the first step the inhomogeneous field of the magnet (operator H_{M}) causes the incoming wavepacket, a hydrogen atom, say, to split into two parts, one part with the eigenvalue $+\frac{1}{2}$, say, of the spin-component operator O, moving upward, and the other part, with the eigenvalue $-\frac{1}{2}$, moving downward.

The spatial separation between the two parts can well reach macroscopic dimensions. In [54], for example, the separation detected on the screen was more than 0.1 mm, even before magnification by means of a microscope. Thus, a deterministic physical law, the Schrödinger equation, connects the path with the spin component [55]. Nevertheless, the two parts of the wavepacket continue to have definite phase relations and to be one and the same wavepacket representing the atom. The atom is therefore neither definitely in the upper nor definitely in the lower path, so that the

'position of the atom' at this stage cannot be the apparatus coordinate indicating the spin component. Note also that each of the, say equal, parts of the wave function that has passed the Stern-Gerlach magnet describes a particle with the parameters of mass m_0, charge e etc., not of $m_0/2$, $e/2$ etc., as would have been expected in a classical hydrodynamic model.

In the second step, provided the criterion proposed in Section 10 is met, the contraction process causes the atomic wave function to contract to a narrow function at the place of one of the two parts while the other part vanishes. This occurs in the screen behind the magnet, the first region where the incoming atom can meet other atoms.

In the third step the narrow function in its interaction with the sensitive cluster initiates the formation of a spot either at the place where the upper or at the place where the lower path arrives on the screen.

The Stern-Gerlach experiment is often called a measurement of the particle's spin component (e.g. [56]), probably because it would indeed be such a measurement if the particle were classical. However, it is not classical, and the spin component deduced from the position of the spot is in general not that of the incoming particle. If we insist that a measurement measures what already *is*, what exactly is measured in the Stern-Gerlach experiment? It depends on what we call a measurement. If we consider the observation of a large number of spots as one single measurement, then the full range of spin components and with this the total spin (and even the absolute value of the magnetic moment) of the particles of a homogeneous ensemble is measured.

Can the observation of only a single spot be considered a measurement? Actually, in the formation of this spot the reducing interaction changes the wavepacket; it forces it to become an eigenpacket of the operator of the spin component along the chosen apparatus axis. Which of the possible eigenpackets it is, is governed by probability laws. Thus we cannot deduce the spin component of the incoming particle by observing a single spot. Nevertheless, observing a spot at a certain position does yield some information about the incoming particle. It can show us whether the particle is a fermion or a boson by the value (in appropriate units) of the distance d of the spot from the symmetry axis: $d = \frac{1}{2}, \frac{3}{2}, \ldots \rightarrow$ fermion, $d = 0, 1, \ldots \rightarrow$ boson. And from $d = \frac{3}{2}$, for instance, we can conclude that the total spin is at least $\frac{3}{2}$. So in some sense we may indeed call the observation of a single spot a measurement, though not of the exact value of the spin, but of some other properties of the incoming particle.

An example of case (b),
concerned with eigenfunctions of a continuous spectrum, is the elastic scattering of a one-particle wavepacket from a fixed potential. This is close to the situation considered in Born's original papers [24], [25]. It may be considered a measurement of the differential scattering cross section (dependence on the deflection angle), or more specifically of a transition probability, since we consider a single incoming wavepacket and not a stationary flux. The cross section is arrived at from a series

of such scattering events. We describe this also as a three-step process:

The first step is the action of the scattering potential (apparatus: H_{M} or U_{M}) on the incoming wavepacket. This then spreads out into an outgoing spherical wave whose amplitude and phase depend on the direction.

The second step is the contraction of the spherical wave to a narrow wavepacket at a cluster in a spherical screen which surrounds the scattering centre, if criterion (10.1) is met at that cluster. This is the first place where the outgoing spherical wave can meet such a cluster. The observable O here is the momentum operator in that particular direction, and the plane waves are its (improper) eigenfunctions. If the energy is high enough, the contracted narrow packet is indeed a good physical representative of the non-physical plane wave. It can be shown that the criterion for this is $\sigma \gg \lambda_0$, where σ is the minimum width in any direction of the packet and λ_0 is its mean de Broglie wavelength. If σ is equal to twice the Bohr radius, $\sigma = 10^{-10}$ m, and λ_0 corresponds to a mean electron energy of 50 keV, that is $\lambda_0 = 5.4 \times 10^{-12}$ m, as in the experiment by Tonomura et al. [12], it is $\sigma/\lambda_0 = 18.5$, and the transverse spreading angle is $\lambda_0/(4\pi\sigma)$ rad $= 0.25\,^\circ$.

The third step is again the formation of a spot in the surrounding screen.

9 The Born rules

We now define what exactly we mean by the Born probability rules. We do not derive them [57]. What we are doing in tis article is to reproduce them in the framework of a deterministic theory. We consider 3 cases:

(i) The probability of finding in a position measurement at time t a value lying in the interval d^3r about $\boldsymbol{r}$ [30, p. 19], [32, p. 117]:

$$\mathrm{d}P_1 = |\psi(\boldsymbol{r},t)|^2 \mathrm{d}^3r, \tag{9.1}$$

where $\psi(\boldsymbol{r},t) \equiv \psi_{\mathrm{i}}$ is the incoming wavepacket.

(ii) The probability of finding in a measurement at time t the eigenvalue o_n of an observable with a *discrete* non-degenerate spectrum [30, p. 216], [57]:

$$P_2 = |(\psi_n(\boldsymbol{r},t), \psi(\boldsymbol{r},t))|^2, \tag{9.2}$$

where $\psi_n(\boldsymbol{r},t)$ is the normalized eigenfunction associated with the eigenvalue o_n.

(iii) The probability of finding at time t an eigenvalue a of an observable with a *continuous* non-degenerate spectrum in the interval $\mathrm{d}a$ about a [30, p. 218]:

$$\mathrm{d}P_3 = |(\psi(a;\boldsymbol{r},t), \psi(\boldsymbol{r},t))|^2 \mathrm{d}a, \tag{9.3}$$

where $\psi(a;\boldsymbol{r},t)$ is the eigenfunction associated with the eigenvalue a.

Note that in the above 3 formulas nothing is said about the situation after the measurement. In particular it is not stated that the final function is the eigenfunction corresponding to the eigenvalue found.

It is important to note that in any case the Born probabilities mean that a certain value *is found*, not that the object measured *has* that value before the measurement.

$\mathrm{d}P_1$ for example means that a position at $\boldsymbol{r}$ *is found*, not that the object *is* at $\boldsymbol{r}$ when it is not observed. This strange conception of the Copenhagen interpretation is an attempt to cope with the purported wave-particle duality. Generally, the probability of finding an object in a certain volume differs from the probability that the object is in that volume. If there is a needle in a haystack the probability of finding it depends on the amount of work and time spent in the search [58].

Consider the normalization convention $\int_{-\infty}^{+\infty} |\psi(\boldsymbol{r},t)|^2 \mathrm{d}^3 r = 1$. Of course, the probability that the object at time t *is* somewhere must be 1, but the probability of *finding* it somewhere at time t may be considerably less than 1. The finding probability can only be 1 if the finding procedure at time t (more on this below) works with an *efficiency* η of 100%.

This shows that in the Copenhagen interpretation formulas (9.1) to (9.3) refer to apparatuses which are tacitly presupposed to function with 100% efficiency. To obtain the efficiency η of a real apparatus (its calibration) requires a large amount of physics and technology. If η is known, the raw data of the measured probabilities (statistical frequencies) can be corrected, that is, divided by η, so that they then correspond to an apparatus with η=1 and refer to the Born probabilities in the Copenhagen interpretation.

The experimenter cannot determine the time t exactly, but he can do so approximately, and in a rather narrow interval on a macroscopic scale. If at some more or less accurately determined time he sends the particle with more or less accurately determined velocity into the apparatus, he may expect to have completed the measurement when the particle has left the apparatus. If the apparatus is 1 m long the time which a 50 keV electron [12] (a 144 m/s atom [52]) spends in it is less than 10^{-8} s (10^{-2} s). 'At time t' practically means within this interval, and an efficiency η=1 of the apparatus means a 100% efficiency of detection within this interval. For all practical purposes there is no real problem here. Note that what is under consideration when we say 'at time t' is the 'absolute' time, not a time interval.

10 The conjectured criterion

We are now ready to formulate the criterion for a contraction to occur. It will depend on two pseudorandom phase constants, α_1 and α_2. α_1 is that of the incoming wavepacket $\psi_{\mathrm{i}}(\boldsymbol{r},t)$, which here already has the form of a special superposition $\psi_{\mathrm{s}}(\boldsymbol{r},t)$ of eigenpackets $\psi_n(\boldsymbol{r},t)$ of an observable O. And α_2 is the phase constant of the wavepacket $\psi_{\mathrm{cl}}(\boldsymbol{r}_0;\boldsymbol{r},t)$ of a small cluster with volume $v_{\mathrm{cl}} \approx \sigma_{\mathrm{clx}}\sigma_{\mathrm{cly}}\sigma_{\mathrm{clz}}$ located about $\boldsymbol{r}_0$ within the screen.

The criterion for a contraction of the incoming normalized wavepacket $\psi_{\mathrm{s}}(\boldsymbol{r},t)$ to occur at some time t and at a cluster located about $\boldsymbol{r}_0$ is then conjectured to be

$$|\alpha_1 - \alpha_2| \leq \tfrac{1}{2}\,\alpha_{\mathrm{S}} \tag{10.1a}$$

and

$$K := \int_{v_{\mathrm{cl}}} |\psi_{\mathrm{s}}(\boldsymbol{r},t)|^2 \,\mathrm{d}^3 r \geq \alpha_2/2\pi, \tag{10.1b}$$

where $\alpha_S = e^2/\hbar c \approx 1/137 \approx 0.00730$ is Sommerfeld's fine-structure constant.

The first part of the criterion, formula (10.1a), is a phase-matching condition. The incoming wavepacket $\psi_s(\boldsymbol{r}, t)$ becomes capable of contraction at the first cluster $\psi_{cl}(\boldsymbol{r}_0; \boldsymbol{r}, t)$ encountered whose absolute phase constant α_2 lies within an interval of size α_S around the absolute phase constant α_1 of $\psi_s(\boldsymbol{r}, t)$. That is, α_2 must lie within the limits $\alpha_1 - \frac{1}{2}\alpha_S \leq \alpha_2 \leq \alpha_1 + \frac{1}{2}\alpha_S$, modulo $\pm 2\pi$.

Given the incoming wavepacket, the position of the phase-matching cluster and the time of the possible contraction are determined by the wave function representing the screen at the moment when the incoming wavepacket arrives. And the screen's wave function is determined by the Schrödinger equation and the contraction processes within the screen which have acted before that moment.

Given that the phase-matching condition is satisfied for a cluster at position $\boldsymbol{r}_0$, the second part of the criterion, condition (10.1b), decides whether or not, at $\boldsymbol{r}_0$ and time t, the contraction will actually occur.

Bearing in mind the argumentation after formula (2.2): any (pseudo)random number, $\alpha_2/2\pi$ (or $\alpha_1/2\pi$), which lies in the interval $[0, K]$ leads to a contraction. The probability that the uniformly distributed random number lies in this interval is just K, so

$$P_4 = K = \int_{v_{cl}} |\psi_s(\boldsymbol{r}, t)|^2 \, \mathrm{d}^3 r \tag{10.2}$$

is the probability that the contraction actually occurs at the phase matching cluster which lies about $\boldsymbol{r}_0$.

As already described in Section 6, the contraction turns the incoming wavepacket into a wavepacket with the spatial dimensions of the cluster and, if it is a sensitive cluster, this wavepacket in turn undergoes a dynamical interaction with the cluster wavepacket.

We point out some general features of the criterion:

1) The phase-matching interval is conjectured to be of the order of Sommerfeld's fine-structure constant α_S because the phases are dimensionless and α_S is the only dimensionless fundamental constant of nature which plays a role in quantum mechanics (cf. hydrogen-like atoms [59], natural linewidth [60]). The smallness of α_S requires the presence of many clusters in the way of the incoming wavepacket for the criterion of a contraction process to be satisfied in an appreciable fraction of cases. The presence of the electric charge e in α_S is regarded as meaning that at least one of the two matching partners must carry an electric charge. This is always the case in the measurement situations under consideration. Note that even neutrons are not thoroughly neutral but have a magnetic moment.

2) The dependence on the phases is contained in the first condition, Eq. (10.1a). The quantity K in the second condition, Eq. (10.1b), is some degree of coverage of the cluster by the absolute square of the wavepacket. No phases are involved in K.

3) The probability P_4 refers only to contraction, at a phase-matching cluster. It does not cover any subsequent dynamical interaction of the contracted wavepacket with the cluster packet. This interaction leads to a spot if the cluster is a *sensitive*

cluster. Even then the spot may not be recorded. Thus to obtain the probability P_5 of recording a spot at the place of a phase-matching cluster, that is, at time t and place $\boldsymbol{r}_0$ we have to multiply the probability P_4 by the overall efficiency $\eta \leq 1$ of spot formation and registration, that is

$$P_5 = \eta\, K = \eta \int_{v_{\rm cl}} |\psi_{\rm s}(\boldsymbol{r}, t)|^2 \, {\rm d}^3 r. \tag{10.3}$$

4) The dependence of the criterion on the phases of both the incoming particle (α_1) and the sensitive cluster in the apparatus (α_2) satisfies Conway and Kochen's [10] definition of contextuality given in the Introduction. It also answers an objection raised by Wigner against deterministic theories [61]. Wigner considers a number of Stern-Gerlach apparatuses in series whose axes point alternately in the z and the x direction, perpendicular to the direction y of the particle entering the respective apparatus: let us assume that the particle in the first apparatus escapes the contracting interaction in the screen with almost unchanged direction and with spin component in the $+z$ direction. Then the value of the determining hidden variable must lie in a fraction of the total range available for it. In the deflection in the subsequent apparatus, with axis in x direction, the value must lie in a fraction of that fraction, and so on; so that after N apparatuses, if N is large enough, it would seem that the hidden variable lies in such a narrow range that it would determine the outcomes of all later experiments. This is in contradiction to the predictions of quantum mechanics. In the present proposal the pseudorandom phases of the wave functions representing the sensitive clusters in the apparatuses ensure the continuing random appearance of the outcomes after any number N of apparatuses.

5) The contextuality here introduced might also be taken as a concrete example of the

> impossibility of any sharp separation between the behaviour of atomic objects and the interaction with the measuring instruments which serve to define the conditions under which the phenomena appear,

as emphasized by Bohr [62].

11 Reproducing the Born rules

We are now going to explain how the conjectured criterion (10.1) leads to the Born probability rules. That is, we are going to show how the probability P_5 of recording a spot at the place of a phase-matching sensitive cluster leads to the Born probability of finding an eigenvalue of an observable. We must take into account that all we can observe are spots emerging from clusters and that the clusters have a small but finite size. Generally, saying that we have observed the position (of anything we have in mind) to be $\boldsymbol{r}_0$ can only mean that we have observed a spot located about $\boldsymbol{r}_0$. The spot achieved may not be much larger than the cluster so that "about $\boldsymbol{r}_0$" means in a region of the linear size $\sigma_{\rm cl}$ of the cluster centered at $\boldsymbol{r}_0$. As stated earlier, $\sigma_{\rm cl}$ is always regarded as small compared to all other relevant distances.

To get from the probability P_5 of recording a spot arising from the sensitive cluster at $\boldsymbol{r}_0$ to the probability P_6 of observing a spot in d^3r we must take into account that the probability P_6 is the product of two probabilities, $P_6 = P_7 \times P_5$.

P_7 is the probability that a phase-matching cluster is met in d^3r. This has nothing to do with the quantum mechanical probabilities under consideration but with our ignorance concerning the detailed structure of the screen, that is, of the exact positions and phase constants of the clusters in it. Since for a position measurement the screen must be homogeneous, the clusters at any rate must be distributed uniformly in it, so that the probability P_7 is indendent of the position of d^3r within the screen.

The preceding section corresponded to the special case $P_7 = 1$. Now we move on to the general case. If there is only one cluster in d^3r then the probability that it is a phase-matching cluster is $P_7 = \alpha_\mathrm{S}/2\pi = 1/861$, namely the ratio of the favourable interval α_S to the total interval 2π. If there are n clusters in d^3r it is

$$P_7 = n\ \alpha_\mathrm{S}/2\pi, \tag{11.1}$$

provided there is no overlap of the intervals α_S belonging to different clusters. In the Appendix it is shown that even with overlap, formula (11.1) is a good approximation in the situations under discussion.

Now, the number n of clusters in d^3r may be written as $\rho\,\mathrm{d}^3r$, where ρ is the cluster density, which is presupposed to be constant because the screen is presupposed to be homogeneous. With this it is $P_7 = \rho\,\mathrm{d}^3r\,\alpha_\mathrm{S}/2\pi$ and with P_5 of (10.3) we obtain

$$P_6 = P_7 \times P_5 = (\eta\,\rho\,\mathrm{d}^3r\,\alpha_\mathrm{S}/2\pi) \int_{v_\mathrm{cl}} |\psi_\mathrm{s}(\boldsymbol{r},t)|^2\,\mathrm{d}^3r. \tag{11.2}$$

For the position measurement we have made it a condition that the sensitive clusters are all the same except that they differ in their absolute phase constants. This, however, is not sufficient to lead to a constant efficiency factor η required for arriving at the Born rules. The efficiency η may depend on $\boldsymbol{r}_0$ because different values of $\boldsymbol{r}_0$ are associated with different eigenvalues of the observable, and the incoming wavepacket ψ_s acts at the place $\boldsymbol{r}_0$ with the properties specified by the associated eigenvalue (Section 6). In the example of the Stern-Gerlach apparatus (Section 8) that part of the incoming wavepacket which lies in the upper path acts as a wavepacket with spin up and that in the lower path as one with spin down. A priori the clusters may react in different ways to spin-up than to spin-down wavepackets. To obtain a constant η, we have to postulate that they react in the same way to the different properties of the incoming wavepacket. This is what is always assumed to be realized in the actual Stern-Gerlach experiments. – Or consider a photograhic emulsion used to measure the relative intensities of spectral lines. Different spectral lines correspond to different momenta or energies, and it cannot always be excluded that the reaction of the sensitive clusters in the emulsion depends on the energy and with this on the position of the cluster involved. In order to obtain a constant η specially prepared

("orthochromatic") emulsions have to be used where there is no such dependence; or where the dependence of the efficiency on energy (= position) can be determined so that the raw data can be appropriately corrected.

As stated in Section 6 the measurements considered are such where the incoming wavepacket ψ_{i} when it meets the screen (in the form of the superposition ψ_{s}) in good approximation is a linear function of space over the region of the cluster. Therefore we can write

$$\int_{v_{\mathrm{cl}}} |\psi_{\mathrm{s}}(\boldsymbol{r},t)|^2 \,\mathrm{d}^3r \approx |\psi_{\mathrm{s}}(\boldsymbol{r}_0,t)|^2 \, v_{\mathrm{cl}}, \tag{11.3}$$

where, as before, $\boldsymbol{r}_0$ denotes the centre of the cluster with volume v_{cl}. Formula (11.2) then assumes the form

$$P_6 = (\eta\rho \, v_{\mathrm{cl}} \, \alpha_{\mathrm{S}}/2\pi) \; |\psi_{\mathrm{s}}(\boldsymbol{r}_0,t)|^2 \,\mathrm{d}^3r. \tag{11.4}$$

The exact values of the factors ($\eta \, \rho \, v_{\mathrm{cl}} \, \alpha_{\mathrm{S}}/2\pi$) in front of $|\psi_{\mathrm{s}}(\boldsymbol{r}_0,t)|^2$ do not matter. We regard their combination as a new efficiency factor ζ, which we finally set equal to 1, just like we did previously with η. If we in addition write $\boldsymbol{r}$ instead of $\boldsymbol{r}_0$ and ψ instead of ψ_{s} we arrive at

$$P_7 = |\psi(\boldsymbol{r},t)|^2 \mathrm{d}^3r,$$

and this is the Born probability $\mathrm{d}P_1$ of formula (9.1).

Going over to quantities different from position we first consider the eigenvalues o_n of a *discrete* spectrum. Any spot observed in the spatial region Δ_n, covered by the eigenfunction ψ_n, means that the eigenvalue o_n has been observed. Thus the probability P_8 of observing o_n is obtained by integrating the probability P_7 of observing a spot in d^3r over the region Δ_n

$$P_8 = \int_{\Delta_n} |\psi(\boldsymbol{r},t)|^2 \mathrm{d}^3r. \tag{11.5}$$

This is like counting the number of spots in Δ_n. We write (11.5) in the form $\int_{\Delta_n} \psi^*(\boldsymbol{r},t)\,\psi(\boldsymbol{r},t)\,\mathrm{d}^3r$ and use the expansion (6.1) $\psi = \sum_n c_n \, \psi_n$. But as we extend the integral $\int_{\Delta_n}$ only over the region covered by ψ_n, and as the apparatus is presupposed to have enough resolving power, so that the distance between neighbouring eigenfunctions is larger that the resolving interval of the apparatus, we can replace the sum $\sum_n$ by one of its terms, so that $\psi \equiv \psi_{\mathrm{s}} = c_n \, \psi_n$. With this we obtain

$$P_8 = |c_n|^2 \int_{\Delta_n} \psi_n^*(\boldsymbol{r},t)\,\psi_n(\boldsymbol{r},t)\,\mathrm{d}^3r, \tag{11.6}$$

and as ψ_n by definition is negligible outside Δ_n the integral may be extended over all space and then is 1 for the normalized ψ_n. With c_n from formula (6.1) we arrive at

$$P_8 = |c_n|^2 = |(\psi_n(\boldsymbol{r},t), \psi(\boldsymbol{r},t))|^2, \tag{11.7}$$

and this is Born's formula (9.2).

In the case of a *continuous* spectrum no apparatus can have sufficient resolving power to separate the eigenvalues from each other, and we can only ask for the probability of observing an eigenvalue in some specified interval Δa (cf. [30, p. 260-265]).

We begin by considering a discrete spectrum whose eigenfunctions ψ_n overlap in space. Having observed a spot in the interval $\Delta^3 r$ then means that we have observed either o_n or o_{n+1} or ... . Thus the probability P_9 of observing an eigenvalue in the interval $\Delta^3 r$ is the sum of the probabilities P_8 over those eigenvalues o_n which lie in the interval $\Delta^3 r$:

$$P_9 = \sum_{o_n \in \Delta^3 r} |(\psi_n(\boldsymbol{r}, t), \psi(\boldsymbol{r}, t))|^2. \tag{11.8}$$

Now, the transition to a continuous spectrum consists in replacing the sum $\sum_{o_n}$ by the integral $\int_a \mathrm{d}a$:

$$P_{10} = \int_a |(\psi(a; \boldsymbol{r}, t), \psi(\boldsymbol{r}, t))|^2 \, \mathrm{d}a, \tag{11.9}$$

where $\psi(a; \boldsymbol{r}, t)$ is the (improper) eigenfunction associated with the eigenvalue a of a continuous spectrum. From this it follows that the probability of observing an eigenvalue in the interval $\mathrm{d}a$ is

$$P_{11} = |(\psi(a; \boldsymbol{r}, t), \psi(\boldsymbol{r}, t)|^2 \, \mathrm{d}a, \tag{11.10}$$

which is Born's formula (9.3).

We may determine the interval $\Delta^3 r$, which belongs to the interval Δa, by considering the spatial support of the eigenfunctions $\psi(a; \boldsymbol{r}, t)$. Note that in doing so we have to consider *physical* (normalizable) eigenfunctions. The mathematical (improper) eigenfunctions of the momentum operator, for example, are plane waves and extend over all space. Physically they have to be replaced by wavepackets, with effectively finite support, that is by superpositions of plane waves.

12 Comparison with the de Broglie-Bohm theory

Finally we will briefly consider some similarities and some differences between our conjecture and the de Broglie-Bohm theory. A full account of the de Broglie-Bohm theory can be found in [5], [6], [8], [37], [63], [64]. Here is a brief summary of the basic postulates (cf. [37, p. 67, 277-281], [63, Sections 8.1, 8.2]):

(1) An individual physical system consists of both a wave propagating in configuration space and a number N of point particles which move continuously under the guidance of the wave.

(2) The wave is mathematically described by $\psi(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N, t) = R(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N, t) \exp(\mathrm{i}S(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N, t)/\hbar)$, a normalized solution of Schrödinger's wave equation.

(3) The position $\boldsymbol{Q}_i(t)$ of the ith point particle of mass m_i is obtained as the solution of the equation

$$\frac{\mathrm{d}\boldsymbol{Q}_i(t)}{\mathrm{d}t} = \frac{1}{m_i}\nabla_{r_i} S(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N, t)\Big|_{\boldsymbol{r}_j = \boldsymbol{Q}_j(t)} \qquad j = 1, \ldots, N.$$

To solve this equation the initial condition $\boldsymbol{Q}_j(t{=}0)$ has to be specified for every j. An ensemble of possible motions associated with the same wave is generated by varying the $\boldsymbol{Q}_j(t{=}0)$.

(4) The probability that at time $t = 0$ particle 1 lies in the volume element d^3Q_1 about the point $\boldsymbol{Q}_1$, particle 2 in d^3Q_2 about $\boldsymbol{Q}_2$, etc. is postulated to be

$$|\psi(\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_N, 0)|^2 \mathrm{d}^3Q_1 \cdots \mathrm{d}^3Q_N.$$

The de Broglie-Bohm theory makes the same predictions as the Copenhagen quantum theory, and it is realistic and deterministic. Yet, despite these advantages it has not found broad acceptance among physicists. I suppose that one of the deepest reasons for this is that its basic element is a point particle. Concerning point particles Einstein wrote [65]:

> On the other hand, it seems to me certain that we have to give up the notion of an absolute localization of the particles in a theoretical model. This seems to me to be the correct theoretical interpretation of Heisenberg's indeterminacy relation.

In the de Broglie-Bohm theory the initial position of the particle is the determining variable [8, p. 97], [37, p. 21, 337, 416, 479], [63, p. 27, 50], [64, p. 137, 144, 145]. In itself it is clearly local. The nonlocal character of the de Broglie-Bohm theory results from the wave function in $3N$-dimensional configuration space, which acts in a nonlocal way because there is only one single time variable. The positions $\{\boldsymbol{Q}_i\}$ of the point particles are additional variables, not contained in the standard formalism of quantum theory. In the Copenhagen interpretation there are also point particles, but they come into existence only in the act of a position measurement. In the de Broglie-Bohm theory the particle is a real physical object, which always exists, regardless of whether or not we measure it. — In our conjecture the determining variable is the (already nonlocal) initial absolute phase. It is already contained in the standard formalism, though it is usually considered to be physically irrelevant and is therefore omitted, as it is in the de Broglie-Bohm theory. On the other hand, in the relativistic case, where we cannot presuppose to have only one time variable, the nonlocal action of the wavepacket is a postulate [22, Section 2.3].

It follows from point (4) above and the Schrödinger equation that the particle positions of the de Broglie-Bohm theory are distributed at any time within the wave according to the density $|\psi(\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_N, t)|^2$ if they are so distributed at the time $t = 0$. In any case the distribution depends on time through the time dependence of the wave functions. — In the present proposal the absolute phase in equilibrium is always uniformly distributed in the interval $[0, 2\pi]$.

We have argued in Section 6 that measurements of every physical quantity can be explained by measurements of positions and times. — This is also so in the de Broglie-Bohm theory [8, p. 10, 34, 166], [37, p. 215], [63, p. 32, 51], where the position is the only property conceded to the particle, whereas all other properties, such as mass, charge, spin, etc. are ascribed to the guiding wave function.

There is no basic reduction in the de Broglie-Bohm theory, but there is an effective reduction (collapse) in that theory because the empty branches of the wave function after a measurement can be disregarded for all practical purposes [64, Section 9.2].

The de Broglie-Bohm theory is based on the Schrödinger equation and is thus nonrelativistic. It is not clear whether or with what modifications it can be extended into the relativistic domain [37, p. 498–509], [63, p. 115], [64, p. 379]. — In our conjecture the multiplication by a constant phase factor is independent of whether it is a nonrelativistic or a relativistic wave function that is multiplied.

In the de Broglie-Bohm theory it is not possible to control the initial positions beyond the $|\psi|^2$ distribution. The fundamental determinism of that theory is thus turned into predictive indeterminism [66]. — The same would have to be said of the present proposal should it turn out that there are situations where the phases cannot be controlled in principle.

Appendix. Phase-interval overlap

We here estimate how much the probability $P_7 = n\,\alpha_S/2\pi$ of formula (11.1) is changed if there is overlap of the phase-constant intervals α_S belonging to different clusters, that is, if the phase constants of neighbouring clusters are not always separated by more than α_S.

P_7 is the ratio of the favourable interval $n\alpha_S$ to the total available interval 2π. In the case of only one cluster, $n = 1$, the favourable interval is α_S. In the case of n clusters without overlap the favourable interval is $n\alpha_S$. In the (hypothetical) extreme opposite case of n clusters with total overlap, that is, where all clusters have the same phase constant, the favourable interval remains α_S as in the case $n = 1$. Of course, due to the quantum nature of the incoming wavepacket, only one of the n clusters can lead to a contraction.

In the case of partial overlap of the n intervals the value of the favourable interval will lie between α_S and $n\,\alpha_S$. We want to calculate the mean value of the favourable interval in many repetitions of the measurement. In order to get an estimate we divide the total interval of 2π into $2\pi/\alpha_S = 861$ sections, each of length α_S, and the number of favourable intervals is the number of intervals which are represented by at least one cluster. We then substitute the overlap problem by an occupancy problem whose solution is known, namely by the birthday problem. In this problem the number 861 of sections corresponds to the 365 days of the year, and the number n of clusters means the number of people in a group. The number of favourable intervals α_S corresponds to the number of birthdays, that is of days which are occupied by at least one person celebrating his or her birthday. This number in turn is the total

number of 365 days minus the number N_0 of empty days (without any birthday). The mean value of the last number for a group of n persons is given in [28, Vol. I, p. 239, 493] as $\bar{N}_0 = 364^n 365^{1-n}$. Thus the mean number $\bar{N}$ of birthdays is $365 - \bar{N}_0$, and the mean fraction of birthdays is $P = (365 - \bar{N}_0)/365$. Returning to the original problem we replace 365 by 861 and obtain

$$\bar{N}_0' = 861 \times (860/861)^n = 861(1 - 1/861)^n = 861 \exp(-c_1 n)$$

with $c_1 = -\ln(1 - 1/861) = 1/860.5$. With this the mean fraction of favourable intervals becomes

$$P_7 = (\bar{N}_0' - 861)/861 = 1 - \exp(-c_1 n) = c_1 n - \frac{1}{2} c_1^2 n^2 \pm \cdots \tag{B1}$$

$$\approx \frac{1}{860.5} n \approx \frac{1}{861} n = \frac{\alpha_S}{2\pi} n \tag{B2}$$

with an error of $\leq 5\%$ for $n = \rho \, \mathrm{d}^3 r \leq 90$. As the chosen physical interval corresponding to the infinitesimal interval $\mathrm{d}^3 r$ can be very small this is an acceptable approximation.

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